An Introduction to Mathematical Statistics and Its Applications

6th Edition

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Cover Image: Glow Green Circle Arcs © Shutterstock/R.T. Wohlstadter

Library of Congress Cataloging-in-Publication Data

Names: Larsen, Richard J. | Marx, Morris L.
Title: An introduction to mathematical statistics and its applications / Richard J. Larsen, Vanderbilt University, Morris L. Marx, University of West Florida.
Description: Sixth edition. | Boston : Pearson, [2018] | Includes bibliographical references and index.
Identifiers: LCCN 2016035117 | ISBN 9780134114217
Subjects: LCSH: Mathematical statistics—Textbooks.
Classification: LCC QA276 .L314 2018 | DDC 519.5—dc23 LC record available at https://lccn.loc.gov/2016035117

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ISBN 13: 978-0-13-411421-7
ISBN 10: 0-13-411421-3

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Preface

John Tukey (1915–2000) was one of the most prominent (and quotable) statisticians of the last half of the twentieth century. He was once asked what he especially enjoyed about the profession he chose for his life's work. "The best thing about being a statistician," he said without hesitation, "is that you get to play in everyone's backyard." That sentiment says much about Tukey's well known eclectic interests; it also speaks to what this book is all about.

Our hope is that this text satisfies two objectives: 1) It introduces the basic techniques of probability and mathematical statistics in a comprehensive and interesting way at a level appropriate for students who have completed three semesters of calculus, and 2) it provides students with the skills and insights necessary to *apply* those principles. In our opinion, satisfying (1) but not (2) would be an inadequate "takeaway" for a student's two-semesters worth of time and effort.

It may seem that completing Objective 1 automatically confers on students the wherewithal to meet Objective 2. Not so. Mathematical statistics deals primarily with the nature of individual measurements or simple properties calculated from samples of measurements—means, variances, distributions, relationships to other measurements, and so on. Analyzing data, though, requires an additional knowledge of the *experimental design* to which an entire set of measurements belong. To borrow some terminology from economists, mathematical statistics is much like the *micro* aspect of the subject; experimental design is the *macro* aspect. There is enough time in a two-semester course to do justice to both.

Experimental designs come in many variations, but *eight* are especially important in terms of the frequency of their occurrence and their relationship to the mathematical statistics covered in a first course. The initial step in teaching someone how to analyze data is helping them learn how to recognize those eight "data models": One-sample data, Two-sample data, k-sample data, Paired data, Randomized block data, Regression/Correlation data, Categorical data, and Factorial data. We believe that mentioning them in passing is not sufficient. They need to be compared and described, *altogether in one chapter*, side-by-side, and illustrated with real-world data.

Identifying data models, of course, is not a difficult skill to acquire. Anyone involved in analyzing data learns it quickly. But for students taking their first course in statistics, ignoring the topic leaves them without a sense of where the subject is going and why. Fully addressing the issue *before* students encounter all the Z tests, t tests, χ^2 tests, and F tests that come in rapid succession provides a very helpful framework for putting all that material in context.

The final step in dealing with Objective 2 is to show the application of mathematical statistics and the methodologies it created to *real-world data.* Made-up or contrived data will not suffice. They do not provide the detail or complexity necessary to point out, for example, why one particular design was used rather than another or what to make of certain anomalies that appear to have occurred or what follow-up studies seem to be warranted. For their help and obvious expertise, we are deeply indebted to all the researchers who have graciously allowed us to use portions of their data to base the more than 80 Case Studies scattered throughout the text. We hope these are as informative and helpful as the "backyards" that Professor Tukey found so enjoyable.

New to This Edition

- Chapter 15, Factorial Data, is a new, downloadable chapter describing the theory and practice of the analysis of variance as it applies to factorial data. It covers two-factor factorials, three-factor factorials, 2^n designs, and fractional factorials, all at the same mathematical level as the book's other two treatments of the analysis of variance, Chapter 12 and Chapter 13. This is the most important of all the multifactor experimental designs.
- Chapter 2 contains ten new examples, including a repeated-independent-trials analysis of the often-quoted "Caesar's last breath" problem.
- Overall, the Sixth Edition contains 18 new Case Studies for additional concept application.
- An Appendix has been added at the end of Chapter 4 summarizing all the important properties of the most frequently used pdfs.
- Much of Section 5.2 dealing with parameter estimation has been rewritten, and the margin of error portion of Section 5.3 has been completely redone.
- Discussions of the different data models in Chapter 8 have been expanded, and an eighth model (factorial data) has been added. The chapter includes seven new Case Studies.
- In Chapter 11, the section on nonlinear models has been thoroughly revised with an emphasis put on their relationship to different laws of growth.
- Because of space and cost considerations, journals and technical reports often display only summaries of an experiment's results. A section has been added to Chapter 12 showing how the entire ANOVA table for a set of k-sample data can be "reconstructed" without knowing any of the individual measurements.
- The text companion website www.pearsonhighered.com/mathstatsresources/ has the online, downloadable Chapter 15 and data sets analyzed in the text, in generic form to copy for input into statistical software. The site also has additional resources to help students and instructors.

Acknowledgments

We would like to thank all of the following reviewers of the present and previous editions of this text for their many helpful comments and suggestions. Every edition benefited from your insight and your expertise.

Reviewers for this edition:
Adam Bowers, University of California, San Diego
Bree Ettinger, Emory University
Eugene D. Gallagher, University of Massachusetts, Boston
Mohammad Kazemi, University of North Carolina, Charlotte
Ralph Russo, University of Iowa
Neslihan Uler, University of Michigan, Ann Arbor
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Reviewers of previous editions:
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Nicolas Christou, University of California, Los Angeles
Daming Xu, University of Oregon
Maria Rizzo, Ohio University
Dimitris Politis, University of California, San Diego

We would also like to express our deepest gratitude to all those on the Pearson team who guided this project to completion. Your professionalism and friendliness made working on this endeavor a pleasure. Thanks to all of you!

What has *not* changed in this edition is our sincere hope that the reader will find mathematical statistics challenging, informative, interesting, and—maybe in a few places here and there when least expected—almost fun.

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Chapter

1 INTRODUCTION

CHAPTER OUTLINE

"Until the phenomena of any branch of knowledge have been submitted to measurement and number it cannot assume the status and dignity of a science."

—Francis Galton

1.1 AN OVERVIEW

Sir Francis Galton was a preeminent biologist of the nineteenth century. A passionate advocate for the theory of evolution (his nickname was "Darwin's bulldog"), Galton was also an early crusader for the study of statistics and believed the subject would play a key role in the advancement of science:

> Some people hate the very name of statistics, but I find them full of beauty and interest. Whenever they are not brutalized, but delicately handled by the higher methods, and are warily interpreted, their power of dealing with complicated phenomena is extraordinary. They are the only tools by which an opening can be cut through the formidable thicket of difficulties that bars the path of those who pursue the Science of man.

Did Galton's prediction come to pass? Absolutely—try reading a biology journal or the analysis of a psychology experiment before taking your first statistics course. Science and statistics have become inseparable, two peas in the same pod. What the good gentleman from London failed to anticipate, though, is the extent to which *all* of us—not just scientists—have become enamored (some would say obsessed) with numerical information. The stock market is awash in averages, indicators, trends, and exchange rates; federal education initiatives have taken standardized testing to new levels of specificity; Hollywood uses sophisticated demographics to see who's watching what, and why; and pollsters regularly tally and track our every opinion, regardless of how irrelevant or uninformed. In short, we have come to expect everything to be measured, evaluated, compared, scaled, ranked, and rated—and if the results are deemed unacceptable for whatever reason, we demand that someone or something be held accountable (in some appropriately quantifiable way).

To be sure, many of these efforts are carefully carried out and make perfectly good sense; unfortunately, others are seriously flawed, and some are just plain nonsense. What they all speak to, though, is the clear and compelling need to know something about the subject of statistics, its uses and its misuses.

This book addresses two broad topics—the *mathematics of statistics* and the *practice of statistics*. The two are quite different. The former refers to the probability theory that supports and justifies the various methods used to analyze data. For the most part, this background material is covered in Chapters 2 through 7. The key result is the *central limit theorem*, which is one of the most elegant and far-reaching results in all of mathematics. (Galton believed the ancient Greeks would have personified and deified the central limit theorem had they known of its existence.) Also included in these chapters is a thorough introduction to combinatorics, the mathematics of systematic counting. Historically, this was the very topic that launched the development of probability in the first place, back in the seventeenth century. In addition to its connection to a variety of statistical procedures, combinatorics is also the basis for every state lottery and every game of chance played with a roulette wheel, a pair of dice, or a deck of cards.

The practice of statistics refers to all the issues (and there are many!) that arise in the design, analysis, and interpretation of data. Discussions of these topics appear in several different formats. Included in most of the case studies throughout the text is a feature entitled "About the Data." These are additional comments about either the particular data in the case study or some related topic suggested by those data. Then near the end of most chapters is a Taking a Second Look at Statistics section. Several of these deal with the *misuses* of statistics—specifically, inferences drawn incorrectly and terminology used inappropriately. The most comprehensive data-related discussion comes in Chapter 8, which is devoted entirely to the critical problem of knowing how to *start* a statistical analysis—that is, knowing which procedure should be used, and why.

More than a century ago, Galton described what he thought a knowledge of statistics should entail. Understanding "the higher methods," he said, was the key to ensuring that data would be "delicately handled" and "warily interpreted." The goal of this book is to help make that happen.

1.2 Some Examples

Statistical methods are often grouped into two broad categories—descriptive statistics and inferential statistics. The former refers to all the various techniques for summarizing and displaying data. These are the familiar bar graphs, pie charts, scatterplots, means, medians, and the like, that we see so often in the print media. The much more mathematical inferential statistics are procedures that make generalizations and draw conclusions of various kinds based on the information contained in a set of data; moreover, they calculate the probability of the generalizations being correct.

Described in this section are three case studies. The first illustrates a very effective use of several descriptive techniques. The latter two illustrate the sorts of questions that inferential procedures can help answer.

CASE STUDY 1.2.1

Pictured at the top of Figure 1.2.1 is the kind of information routinely recorded by a seismograph—listed chronologically are the occurrence times and Richter magnitudes for a series of earthquakes. As raw data, the numbers are largely meaningless: No patterns are evident, nor is there any obvious connection between the frequencies of tremors and their severities.

Shown at the bottom of the figure is the result of applying several descriptive techniques to an actual set of seismograph data recorded over a period of several years in southern California (73). Plotted above the Richter (R) value of 4.0, for example, is the average number (N) of earthquakes occurring per year in that region having magnitudes in the range 3.75 to 4.25. Similar points are

Episode number	Date	Time	Severity (Richter scale)
⋮	⋮	⋮	⋮
217	6/19	4:53 P.M.	2.7
218	7/2	6:07 A.M.	3.1
219	7/4	8:19 A.M.	2.0
220	8/7	1:10 A.M.	4.1
221	8/7	10:46 P.M.	3.6
⋮	⋮	⋮	⋮

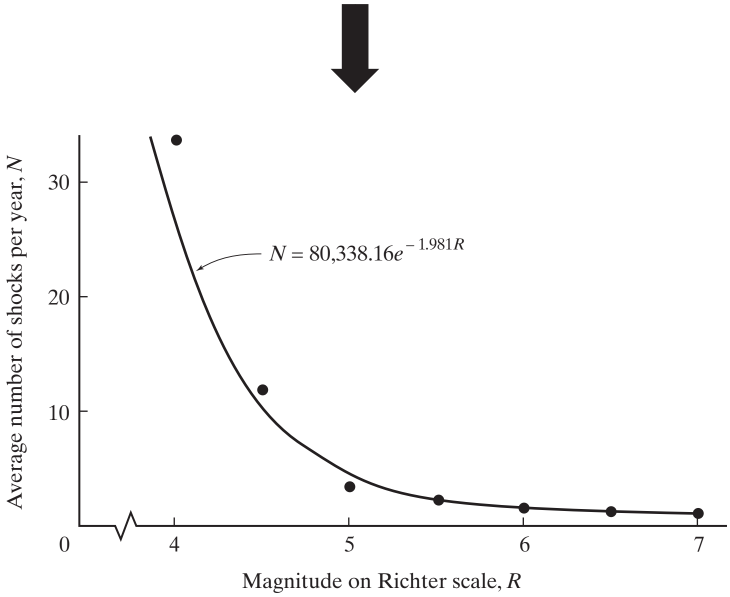


Figure 1.2.1

included for R-values centered at 4.5, 5.0, 5.5, 6.0, 6.5, and 7.0. Now we can see that earthquake frequencies and severities are clearly related: Describing the (N, R)'s exceptionally well is the equation

$$N = 80{,}338.16e^{-1.981R} \tag{1.2.1}$$

which is found using a procedure described in Chapter 11. (*Note:* Geologists have shown that the model $N = \beta_0 e^{\beta_1 R}$ describes the (N, R) relationship all over the world. All that changes from region to region are the numerical values for β_0 and β_1.)

Notice that Equation 1.2.1 is more than just an elegant summary of the observed (N, R) relationship. It also allows us to estimate the likelihood of future earthquake catastrophes having values of R that have never been observed. On the minds of all Californians, of course, is the Big One, the dreaded rip-roaring 10.5-Richter-scale monster megaquake that turns buildings into piles of rubble, sends busloads of tourists careening into the San Francisco Bay, and moves the intersection of Hollywood and Vine to somewhere in downtown El Segundo. How often might an earthquake of that magnitude be expected?

Letting $R = 10.5$ in Equation 1.2.1 gives

$$\begin{aligned} N &= 80{,}338.16e^{-1.981(10.5)} \\ &= 0.0000752 \text{ earthquakes/year} \end{aligned}$$

(Continued on next page)

(Case Study 1.2.1 continued)

so the predicted frequency would be once every 13,300 years ($= 1/0.0000752$). On the one hand, the rarity of such a disaster has to be reassuring; on the other hand, not knowing when the *last* such earthquake occurred is a bit unsettling. Are we on borrowed time? What if the most recent megaquake occurred forty thousand years ago?

Comment The megaquake prediction prompted by Equation 1.2.1 raises an obvious question: Why is the calculation that led to the model $N = 80{,}338.16e^{-1.981R}$ not considered an example of inferential statistics even though it did yield a prediction for $R = 10.5$? The answer is that Equation 1.2.1—by itself—does not tell us anything about the "error" associated with its predictions. In Chapter 11, a more elaborate probability method based on Equation 1.2.1 is described that does yield error estimates and qualifies as a bona fide inference procedure.

About the Data For the record, the strongest earthquake ever recorded in California occurred January 9, 1857 along the San Andreas fault near Fort Tejon, a sparsely populated settlement about seventy miles north of Los Angeles. Its magnitude was estimated to be between 7.9 and 8.0. The state's most famous, deadliest, and costliest earthquake, though, occurred in San Francisco on April 18, 1906. It had a magnitude of 7.8, killed three thousand people, and destroyed 80% of the city. Over the years, a number of Hollywood movies featured that particular earthquake as part of their storylines, the best known being *San Francisco*, a 1936 production starring Clark Gable and Spencer Tracy.

CASE STUDY 1.2.2

In folklore, the full moon is often portrayed as something sinister, a kind of evil force possessing the power to control our behavior. Over the centuries, many prominent writers and philosophers have shared this belief. Milton, in *Paradise Lost*, refers to

> Demoniac frenzy, moping melancholy
> And moon-struck madness.

And Othello, after the murder of Desdemona, laments

> It is the very error of the moon,
> She comes more near the earth than she was wont
> And makes men mad.

On a more scholarly level, Sir William Blackstone, the renowned eighteenth-century English barrister, defined a "lunatic" as

> one who hath ... lost the use of his reason and who hath lucid intervals, sometimes enjoying his senses and sometimes not, and that frequently depending upon changes of the moon.

The possibility of lunar phases influencing human affairs is a theory not without supporters among the scientific community. Studies by reputable medical researchers have attempted to link the "Transylvania effect," as it has come to be known, with higher suicide rates, pyromania, and epilepsy (not to mention the periodic unseemly behavior of werewolves ...).

The possible relationship between lunar cycles and mental breakdowns has also been studied. Table 1.2.1 shows one year's admission rates to the emergency room of a Virginia mental health clinic *before, during, and after* its twelve full moons (13).

Table 1.2.1 Admission Rates (Patients/Day)

Month	Before Full Moon	During Full Moon	After Full Moon
Aug.	6.4	5.0	5.8
Sept.	7.1	13.0	9.2
Oct.	6.5	14.0	7.9
Nov.	8.6	12.0	7.7
Dec.	8.1	6.0	11.0
Jan.	10.4	9.0	12.9
Feb.	11.5	13.0	13.5
Mar.	13.8	16.0	13.1
Apr.	15.4	25.0	15.8
May	15.7	13.0	13.3
June	11.7	14.0	12.8
July	15.8	20.0	14.5
Averages:	10.9	13.3	11.5

Notice for these data that the average admission rate "during" the full moon is, in fact, *higher* than the "before" and "after" admission rates: 13.3 as opposed to 10.9 and 11.5. Can it be inferred, then, *from these averages* that the data support the existence of a Transylvania effect? No. Another explanation for the averages is possible—namely, that there is no such thing as a Transylvania effect and the observed differences among the three averages are due solely to chance.

Questions of this sort—that is, choosing between two conflicting explanations for a set of data—are resolved using a variety of techniques known as *hypothesis testing*. The insights that hypothesis tests provide are without question the most important contribution that the subject of statistics makes to the advancement of science.

The particular hypothesis test appropriate for the data in Table 1.2.1 is known as a *randomized block analysis of variance*, which will be covered at length in Chapter 13. As we will see, the conclusion reached in this case is both unexpected and a bit disconcerting.

CASE STUDY 1.2.3

It may not be made into a movie anytime soon, but the way that statistical inference was used to spy on the Nazis in World War II is a pretty good tale. And it certainly did have a surprise ending!

The story began in the early 1940s. Fighting in the European theatre was intensifying, and Allied commanders were amassing a sizeable collection of abandoned and surrendered German weapons. When they inspected those weapons, the Allies noticed that each one bore a different number. Aware of the Nazis' reputation for detailed record keeping, the Allies surmised that each number represented the chronological order in which the piece had been manufactured. But if that was true, might it be possible to use the "captured" serial numbers to estimate the total number of weapons the Germans had produced?

That was precisely the question posed to a group of government statisticians working out of Washington, D.C. Wanting to estimate an adversary's manufacturing capability was, of course, nothing new. Up to that point, though, the only sources of that information had been spies and traitors; using serial numbers was an approach entirely different.

(Continued on next page)

(Case Study 1.2.3 continued)

The answer turned out to be a fairly straightforward application of the principles that will be introduced in Chapter 5. If n is the total number of captured serial numbers and $x_{\max}$ is the largest captured serial number, then the estimate for the total number of items produced is given by the formula

$$\text{estimated output} = [(n+1)/n]x_{\max} - 1 \tag{1.2.2}$$

Suppose, for example, that $n = 5$ tanks were captured and they bore the serial numbers 92, 14, 28, 300, and 146, respectively. Then $x_{\max} = 300$ and the estimated total number of tanks manufactured is three hundred fifty-nine:

$$\begin{aligned}\text{estimated output} &= [(5+1)/5]300 - 1 \\ &= 359\end{aligned}$$

Did Equation 1.2.2 work? Better than anyone could have expected (probably even the statisticians). When the war ended and the Third Reich's "true" production figures were revealed, it was found that serial number estimates were far more accurate in every instance than all the information gleaned from traditional espionage operations, spies, and informants. The serial number estimate for German tank production in 1942, for example, was 3400, a figure very close to the actual output. The "official" estimate, on the other hand, based on intelligence gathered in the usual ways, was a grossly inflated 18,000.

About the Data Large discrepancies, like 3400 versus 18,000 for the tank estimates, were not uncommon. The espionage-based estimates were consistently erring on the high side because of the sophisticated Nazi propaganda machine that deliberately exaggerated the country's industrial prowess. On spies and would-be adversaries, the Third Reich's carefully orchestrated dissembling worked exactly as planned; on Equation 1.2.2, though, it had no effect whatsoever! (69)

1.3 A Brief History

For those interested in how we managed to get to where we are, Section 1.3 offers a brief history of probability and statistics. The two subjects were not mathematical littermates—they began at different times in different places by different people for different reasons. How and why they eventually came together makes for an interesting story, reacquaints us with some towering figures from the past, and introduces several others whose names will probably not be familiar but whose contributions were critically important in demonstrating the linkage between science and statistics.

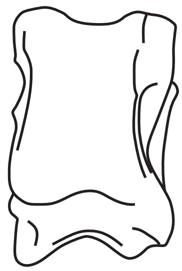
PROBABILITY: THE EARLY YEARS

No one knows where or when the notion of chance first arose; it fades into our prehistory. Nevertheless, evidence linking early humans with devices for generating random events is plentiful: Archaeological digs, for example, throughout the ancient world consistently turn up a curious overabundance of *astragali*, the heel bones of sheep and other vertebrates. Why should the frequencies of these bones be so disproportionately high? One could hypothesize that our forebears were fanatical foot fetishists, but two other explanations seem more plausible: The bones were used for religious ceremonies *and for gambling*.

Astragali have six sides but are not symmetrical (see Figure 1.3.1). Those found in excavations typically have their sides numbered or engraved. For many ancient civilizations, astragali were the primary mechanism through which oracles solicited the opinions of their gods. In Asia Minor, for example, it was customary in divination rites to roll, or *cast*, five astragali. Each possible configuration was associated with the name of a god and carried with it the sought-after advice. An outcome of (1, 3, 3, 4, 4), for instance, was said to be the throw of the savior Zeus, and its appearance was taken as a sign of encouragement (37):

> One one, two threes, two fours
> The deed which thou meditatest, go do it boldly.
> Put thy hand to it. The gods have given thee
> favorable omens
> Shrink not from it in thy mind, for no evil
> shall befall thee.

Figure 1.3.1



Sheep astragalus

A (4, 4, 4, 6, 6), on the other hand, the throw of the child-eating Cronos, would send everyone scurrying for cover:

> Three fours and two sixes. God speaks as follows.
> Abide in thy house, nor go elsewhere,
> Lest a ravening and destroying beast come nigh thee.
> For I see not that this business is safe. But bide
> thy time.

Gradually, over thousands of years, astragali were replaced by dice, and the latter became the most common means for generating random events. Pottery dice have been found in Egyptian tombs built before 2000 B.C.; by the time the Greek civilization was in full flower, dice were everywhere. (*Loaded* dice have also been found. Mastering the mathematics of probability would prove to be a formidable task for our ancestors, but they seemed quite adept at learning how to cheat!)

The lack of historical records blurs the distinction initially drawn between divination ceremonies and recreational gaming. Among more recent societies, though, gambling emerged as a distinct entity, and its popularity was irrefutable. The Greeks and Romans were consummate gamblers, as were the early Christians (99).

Rules for many of the Greek and Roman games have been lost, but we can recognize the lineage of certain modern diversions in what was played during the Middle Ages. The most popular dice game of that period was called *hazard*, the name deriving from the Arabic *al zhar*, which means "a die." Hazard is thought to have been brought to Europe by soldiers returning from the Crusades; its rules are much like those of our modern-day craps. Cards were first introduced in the fourteenth century and immediately gave rise to a game known as *Primero*, an early form of poker. Board games such as backgammon were also popular during this period.

Given this rich tapestry of games and the obsession with gambling that characterized so much of the Western world, it may seem more than a little puzzling that a formal study of probability was not undertaken sooner than it was. As we will see shortly, the first instance of anyone *conceptualizing* probability in terms of a mathematical model occurred in the sixteenth century. That means that more than two thousand years of dice games, card games, and board games passed by before someone finally had the insight to write down even the simplest of probabilistic abstractions.

Historians generally agree that, as a subject, probability got off to a rocky start because of its incompatibility with two of the most dominant forces in the evolution of our Western culture, Greek philosophy and early Christian theology. The Greeks were comfortable with the notion of chance (something the Christians were not), but it went against their nature to suppose that random events could be quantified in any useful fashion. They believed that any attempt to reconcile mathematically what *did* happen with what *should have* happened was, in their phraseology, an improper juxtaposition of the "earthly plane" with the "heavenly plane."

Making matters worse was the antiempiricism that permeated Greek thinking. Knowledge, to them, was not something that should be derived by experimentation. It was better to reason out a question logically than to search for its explanation in a set of numerical observations. Together, these two attitudes had a deadening effect: The Greeks had no motivation to think about probability in any abstract sense, nor were they faced with the problems of interpreting data that might have pointed them in the direction of a probability calculus.

If the prospects for the study of probability were dim under the Greeks, they became even worse when Christianity broadened its sphere of influence. The Greeks and Romans at least accepted the *existence* of chance. However, they believed their gods to be either unable or unwilling to get involved in matters so mundane as the outcome of the roll of a die. Cicero writes:

> Nothing is so uncertain as a cast of dice, and yet there is no one who plays often who does not make a Venus-throw[1] and occasionally twice and thrice in succession. Then are we, like fools, to prefer to say that it happened by the direction of Venus rather than by chance?

For the early Christians, though, there was no such thing as chance: Every event that happened, no matter how trivial, was perceived to be a direct manifestation of God's deliberate intervention. In the words of St. Augustine:

> Nos eas causas quae dicuntur fortuitae . . . non dicimus
> nullas, sed latentes; easque tribuimus vel veri Dei . . .
> (We say that those causes that are said to be by chance
> are not non-existent but are hidden, and we attribute
> them to the will of the true God . . .)

Taking Augustine's position makes the study of probability moot, and it makes a probabilist a heretic. Not surprisingly, nothing of significance was accomplished in the subject for the next fifteen hundred years.

It was in the sixteenth century that probability, like a mathematical Lazarus, arose from the dead. Orchestrating its resurrection was one of the most eccentric figures in the entire history of mathematics, Gerolamo Cardano. By his own admission, Cardano personified the best and the worst—the Jekyll and the Hyde—of the Renaissance man. He was born in 1501 in Pavia. Facts about his personal life

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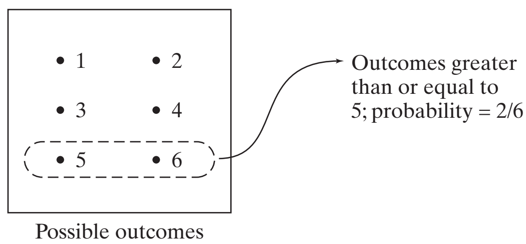
[1] When rolling four astragali, each of which is numbered on *four* sides, a Venus-throw was having each of the four numbers appear.

are difficult to verify. He wrote an autobiography, but his penchant for lying raises doubts about much of what he says. Whether true or not, though, his "one-sentence" self-assessment paints an interesting portrait (135):

> Nature has made me capable in all manual work, it has given me the spirit of a philosopher and ability in the sciences, taste and good manners, voluptuousness, gaiety, it has made me pious, faithful, fond of wisdom, meditative, inventive, courageous, fond of learning and teaching, eager to equal the best, to discover new things and make independent progress, of modest character, a student of medicine, interested in curiosities and discoveries, cunning, crafty, sarcastic, an initiate in the mysterious lore, industrious, diligent, ingenious, living only from day to day, impertinent, contemptuous of religion, grudging, envious, sad, treacherous, magician and sorcerer, miserable, hateful, lascivious, obscene, lying, obsequious, fond of the prattle of old men, changeable, irresolute, indecent, fond of women, quarrelsome, and because of the conflicts between my nature and soul I am not understood even by those with whom I associate most frequently.

Formally trained in medicine, Cardano's interest in probability derived from his addiction to gambling. His love of dice and cards was so all-consuming that he is said to have once sold all his wife's possessions just to get table stakes! Fortunately, something positive came out of Cardano's obsession. He began looking for a mathematical model that would describe, in some abstract way, the outcome of a random event. What he eventually formalized is now called the *classical definition of probability*: If the total number of possible outcomes, all equally likely, associated with some action is n, and if m of those n result in the occurrence of some given event, then the probability of that event is m/n. If a fair die is rolled, there are $n = 6$ possible outcomes. If the event "outcome is greater than or equal to 5" is the one in which we are interested, then $m = 2$ (the outcomes 5 and 6) and the probability of the event is $\frac{2}{6}$, or $\frac{1}{3}$ (see Figure 1.3.2).

Figure 1.3.2



Cardano had tapped into the most basic principle in probability. The model he discovered may seem trivial in retrospect, but it represented a giant step forward: His was the first recorded instance of anyone computing a *theoretical*, as opposed to an empirical, probability. Still, the actual impact of Cardano's work was minimal. He wrote a book in 1525, but its publication was delayed until 1663. By then, the focus of the Renaissance, as well as interest in probability, had shifted from Italy to France.

The date cited by many historians (those who are not Cardano supporters) as the "beginning" of probability is 1654. In Paris, a well-to-do gambler the Chevalier de Méré asked several prominent mathematicians, including Blaise Pascal, a series of questions, the best known of which is the *problem of points*:

> Two people, A and B, agree to play a series of fair games until one person has won six games. They each have wagered the same amount of money, the intention being that the winner will be awarded the entire pot. But suppose, for whatever reason, the series is prematurely terminated, at which point A has won five games and B three. How should the stakes be divided?